

## NOTES ON STABLE TEICHMÜLLER QUASIGEODESICS

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ABSTRACT. In this note, we prove that for a cobounded, Lipschitz path  $\gamma : I \rightarrow \mathcal{T}$  if the pull back bundle  $\mathcal{H}_\gamma$  over  $I$  is a strongly relatively hyperbolic metric space then there exists a geodesic  $\xi$  in  $\mathcal{T}$  such that  $\gamma(I)$  and  $\xi$  are close to each other.

Suppose  $S_{g,n}$  is a surface of genus  $g$  with  $n$  punctures such that its Euler characteristic  $\chi(S_{g,n}) < 0$ . Consider the Teichmüller space  $\mathcal{T} = \text{Teich}(S_{g,n})$  of  $S_{g,n}$ , there is a smooth fiber bundle  $\mathcal{S} \rightarrow \mathcal{T}$  over  $\mathcal{T}$ , whose fiber  $\mathcal{S}_\sigma$  over  $\sigma \in \mathcal{T}$  is  $S_{g,n}$  with metric  $\sigma$ . Let  $\mathcal{H}$  be the universal cover of  $\mathcal{S}$ , then the universal covering  $\mathcal{H} \rightarrow \mathcal{S}$  defines a smooth fiber bundle  $\mathcal{H} \rightarrow \mathcal{T}$  whose fiber  $\mathcal{H}_\sigma$  over  $\sigma \in \mathcal{T}$  is isometric to the hyperbolic plane  $\mathbb{H}^2$ . The purpose of this note is to prove that for a  $\mathcal{B}$ -cobounded, Lipschitz path  $\gamma : I \rightarrow \mathcal{T}$ , where  $\mathcal{B}$  is a compact subset of  $\mathcal{T}$ , if the pull back bundle  $\mathcal{H}_\gamma$  over  $I$  is a strongly relatively hyperbolic metric space then there exists a geodesic  $\xi$  in  $\mathcal{T}$  such that the Hausdorff distance between  $\gamma(I)$  and  $\xi$  is bounded. This is a straightforward generalization of a result due to Mosher, Theorem 1.1 of [9], where the statement was proven for closed surfaces admitting hyperbolic metrics with the assumption that  $\mathcal{H}_\gamma$  is a hyperbolic metric space.

## 1. RELATIVE HYPERBOLICITY

Let  $X$  be a path metric space. A collection of closed subsets  $\mathcal{D} = \{D_\alpha\}$  of  $X$  will be said to be **uniformly separated** if there exists  $\epsilon > 0$  such that  $d(D_1, D_2) \geq \epsilon$  for all distinct  $D_1, D_2 \in \mathcal{D}$ .

**Definition 1.1.** (Farb [4]) The **electric space** (or **coned-off space**)  $\mathcal{E}(X, \mathcal{D})$  corresponding to the pair  $(X, \mathcal{D})$  is a metric space which consists of  $X$  and a collection of vertices  $v_\alpha$  (one for each  $D_\alpha \in \mathcal{D}$ ) such that each point of  $D_\alpha$  is joined to (coned off at)  $v_\alpha$  by an edge of length  $\frac{1}{2}$ .  $X$  is said to be **weakly hyperbolic** relative to the collection  $\mathcal{D}$  if  $\mathcal{E}(X, \mathcal{D})$  is a hyperbolic metric space.

For a path  $\gamma \subset X$ , there is an induced path  $\hat{\gamma}$  in  $\mathcal{E}(X, \mathcal{D})$  obtained by coning the portions of  $\gamma$  lying in sets  $D \in \mathcal{D}$ . If  $\hat{\gamma}$  is a geodesic (resp.  $P$ -quasigeodesic) in  $\mathcal{E}(X, \mathcal{D})$ ,  $\gamma$  is called a *relative geodesic* (resp. *relative  $P$ -quasigeodesic*).

**Definition 1.2.** [2] *Relative geodesics (resp.  $P$ -quasigeodesics) in  $(X, \mathcal{D})$  are said to satisfy bounded region penetration properties if there exists  $K = K(P) > 0$  such that for any two relative geodesics (resp.  $P$ -quasigeodesics without backtracking)  $\beta, \gamma$  joining  $x, y \in X$  following two properties are satisfied:*

- (1) *if precisely one of  $\{\beta, \gamma\}$  meets a set  $D_\alpha$ , then the length (measured in the intrinsic path-metric on  $D_\alpha$ ) from the first (entry) point to the last (exit) point (of the relevant path) is at most  $K$ ,*
- (2) *if both  $\{\beta, \gamma\}$  meet some  $D_\alpha$  then the length (measured in the intrinsic path-metric on  $D_\alpha$ ) from the entry point of  $\beta$  to that of  $\gamma$  is at most  $K$ ; similarly for exit points.*

**Definition 1.3.** (Farb [2])  $X$  is said to be *hyperbolic relative to the uniformly separated collection  $\mathcal{D}$*  if  $X$  is weakly hyperbolic relative to  $\mathcal{D}$  and relative  $P$  quasigeodesics without backtracking satisfy the bounded region penetration properties.

Gromov's definition of relative hyperbolicity :

**Definition 1.4.** [7] *For any geodesic metric space  $(D, d)$ , the hyperbolic cone (analog of a horoball)  $D^h$  is the metric space  $D \times [0, \infty) = D^h$  equipped with the path metric  $d_h$  obtained from two pieces*

of data

1)  $d_{h,t}((x,t), (y,t)) = 2^{-t}d_D(x,y)$ , where  $d_{h,t}$  is the induced path metric on  $D_t = D \times \{t\}$ . Paths joining  $(x,t), (y,t)$  and lying on  $D_t = D \times \{t\}$  are called horizontal paths.

2)  $d_h((x,t), (x,s)) = |t-s|$  for all  $x \in D$  and for all  $t, s \in [0, \infty)$ , and the corresponding paths are called vertical paths.

3) for all  $x, y \in D^h$ ,  $d_h(x,y)$  is the path metric induced by the collection of horizontal and vertical paths.

**Definition 1.5.** [7] Let  $\delta \geq 0$ . Let  $X$  be a geodesic metric space and  $\mathcal{D}$  be a collection of mutually disjoint uniformly separated subsets of  $X$ .  $X$  is said to be  $\delta$ -hyperbolic relative to  $\mathcal{D}$  in the sense of Gromov, if the quotient space  $\mathcal{G}(X, \mathcal{D})$ , obtained by attaching the hyperbolic cones  $D^h$  to  $D \in \mathcal{D}$  via the identification  $(x,0) \sim x$  for all  $x \in D$ , is a  $\delta$ -hyperbolic metric space.  $X$  is said to be hyperbolic relative to  $\mathcal{D}$  in the sense of Gromov if  $\mathcal{G}(X, \mathcal{D})$  is a  $\delta$ -hyperbolic metric space for some  $\delta \geq 0$ .

**Theorem 1.6.** (Bowditch [1]) Let  $X$  be a geodesic metric space and  $\mathcal{D}$  be a collection of mutually disjoint uniformly separated subsets of  $X$ .  $X$  is hyperbolic relative to the collection  $\mathcal{D}$  of uniformly separated subsets of  $X$  in the sense of Farb if and only if  $X$  is hyperbolic relative to the collection  $\mathcal{D}$  of uniformly separated subsets of  $X$  in the sense of Gromov.

## 2. MAIN THEOREM

Suppose  $p_1, \dots, p_n$  are the punctures of  $S_{g,n}$ , then each Teichmüller metric  $\sigma$  on  $S_{g,n}$  corresponds to collections  $\mathcal{D}_\sigma(p_1), \dots, \mathcal{D}_\sigma(p_n)$  of horodisks in the fiber  $\mathcal{H}_\sigma$  of the bundle  $\mathcal{H} \rightarrow \mathcal{T}$  satisfying the following properties:

- (1) let  $\mathcal{D}_\sigma(p_i) = \{D_\sigma(p_i, \alpha) : \alpha \in \Lambda\}$ , then for each  $i$  and  $\alpha$  there exists a sub-bundle  $\mathcal{D}(p_i, \alpha) \rightarrow \mathcal{T}$  such that the fiber over  $\sigma \in \mathcal{T}$  is  $D_\sigma(p_i, \alpha)$ .
- (2) each  $\mathcal{D}_\sigma(p_i)$  is invariant under the action of  $\pi_1(S_{g,n})$ ,
- (3) elements of  $\mathcal{D}_\sigma(p_1) \cup \dots \cup \mathcal{D}_\sigma(p_n)$  are disjoint with each other,

For each path  $\gamma : I \rightarrow \mathcal{T}$ ,  $1 \leq i \leq n$  and  $\alpha \in \Lambda$ , there exists a pull back bundle  $\mathcal{D}_\gamma(p_i, \alpha) \rightarrow I$  such that the fiber over  $t \in I$  is  $D_{\gamma(t)}(p_i, \alpha)$ . Let  $\mathcal{D}_\gamma$  denote the collection  $\{\mathcal{D}_\gamma(p_i, \alpha) : 1 \leq i \leq n, \alpha \in \Lambda\}$ . Consider a subset  $\mathcal{B}$  of the moduli space  $\mathcal{M} = \mathcal{T}/MCG(S_{g,n})$ , a path  $\gamma : I \rightarrow \mathcal{T}$  is said to be  $\mathcal{B}$ -cobounded, if the image of  $\gamma$  under the projection  $\mathcal{T} \rightarrow \mathcal{M}$  lies in  $\mathcal{B}$ . We prove the following theorem:

**Theorem 2.1.** Let  $I$  be a closed, connected interval of  $\mathbb{R}$ . For a compact subset  $\mathcal{B}$  of the moduli space  $\mathcal{M} = \mathcal{T}/MCG(S_{g,n})$  and for every  $\rho \geq 1, \delta \geq 0$  there exists  $P \geq 0$  such that the following holds:

If  $\gamma : I \rightarrow \mathcal{T}$  is  $\mathcal{B}$ -cobounded and  $\rho$ -Lipschitz path, and if  $\mathcal{H}_\gamma$  is strongly  $\delta$ -hyperbolic relative to the collection  $\mathcal{D}_\gamma$ , then there exists a geodesic  $\xi : I \rightarrow \mathcal{T}$  joining end points of  $\gamma$  such that the Hausdorff distance between  $\gamma(I)$  and  $\xi(I)$  is at most  $P$ .

Note that the fibers  $\mathcal{H}_\sigma = \mathbb{H}^2 \times \sigma$  of  $\mathcal{H} \rightarrow \mathcal{T}$  are (uniformly) strongly hyperbolic relative to the collections  $\mathcal{D}_\sigma = \{D_\sigma(p_i, \alpha) : 1 \leq i \leq n, \alpha \in \Lambda\}$  of horodisks. Hence the coned-off spaces  $\mathcal{E}(\mathcal{H}_\sigma, \mathcal{D}_\sigma)$ ,  $\sigma \in \mathcal{T}$ , are (uniformly) hyperbolic metric spaces. Thus for a path  $\gamma : I \rightarrow \mathcal{T}$ , there exists a bundle  $\mathcal{PH}_\gamma \rightarrow I$  of coned-off hyperbolic metric spaces with fiber  $\mathcal{E}(\mathcal{H}_{\gamma(t)}, \mathcal{D}_{\gamma(t)})$ .  $\mathcal{PH}_\gamma$  is also obtained by partially electrocuting each element  $\mathcal{D}_\gamma(p_i, \alpha)$  of  $\mathcal{D}_\gamma$  to a hyperbolic space  $\mathcal{L}_\gamma(p_i, \alpha)$ , where  $\mathcal{L}_\gamma(p_i, \alpha)$  is the locus of cone points obtained by coning  $D_{\gamma(t)}(p_i, \alpha)$  for all  $t \in I$ . By Lemma 2.8 of [6], if  $\mathcal{H}_\gamma$  is strongly hyperbolic relative to the collection  $\mathcal{D}_\gamma$  then  $\mathcal{PH}_\gamma$  is a hyperbolic metric space.

**Definition 2.2.** Given  $\kappa > 1$ , a natural number  $n$ ,  $A \geq 0$ , a sequence of positive numbers  $\{r_j : j \in J\}$ , where  $J$  is a subinterval of set of integers  $\mathbb{Z}$ , is said to satisfy  $(\kappa, n, A)$ -flaring property if  $j-n, j+n \in J$  and if  $r_j > A$  then  $\max\{r_{j-n}, r_{j+n}\} \geq \kappa r_j$ .

A path  $\alpha : J \rightarrow \mathcal{PH}_\gamma$ , where  $J \subset I$ , is said to be  $\lambda$ -quasivertical if it is  $\lambda$ -Lipschitz and also a section. Let  $d_\sigma^\wedge$  denote the metric of the coned-off space  $\mathcal{E}(\mathcal{H}_\sigma, \mathcal{D}_\sigma)$ . Since  $\mathcal{PH}_\gamma$  is a hyperbolic space, so we have the following flaring properties:

**Proposition 2.3.** (Theorem 4.7 of [6]) *With the notations as above, given  $\lambda \geq 1$  there exist  $\kappa > 1$ , an integer  $n \geq 1$  and a number  $A > 0$  such that the following holds:*

*Let  $\alpha, \beta : J \rightarrow \mathcal{PH}_\gamma$  be two  $\lambda$ -quasivertical paths, then the sequence  $s_j = d_{\widehat{(\gamma(j))}}(\alpha(j), \beta(j))$ , where  $j \in J \cap \mathbb{Z}$ , satisfies  $(\kappa, n, A)$ -flaring property.*

We refer to [3] for the definitions of measured foliation  $\mathcal{MF}$  and measured geodesic lamination  $\mathcal{MGL}$  of general hyperbolic surfaces. For each  $\mu \in \mathcal{MF}$ , let  $\mu_t$  denote the measured geodesic lamination on the hyperbolic surface  $\mathcal{S}_{\gamma(t)} = \mathcal{H}_{\gamma(t)}/\pi_1(S_{g,n})$ . Let  $\mathcal{S}_{\gamma(t)}^b$  denote the ‘thick part’ of  $\mathcal{S}_{\gamma(t)}$  i.e.  $\mathcal{S}_{\gamma(t)}^b$  is obtained from  $\mathcal{S}_{\gamma(t)}$  by deleting the images of interior of horodisks under the projection  $\mathcal{H}_{\gamma(t)} \rightarrow \mathcal{S}_{\gamma(t)}$ . Now each  $\mu \in \mathcal{MF}$  induce a geodesic lamination  $\mu_t^b (\subset \mu_t)$  on  $\mathcal{S}_{\gamma(t)}^b$ . A connection path of the sub-bundle  $\mathcal{S}_\gamma^b \rightarrow I$  is a piecewise smooth section of the projection map which is everywhere tangent to the connection on the bundle  $\mathcal{S}_\gamma^b \rightarrow I$ . The connection map  $h_{st} : \mathcal{S}_{\gamma(s)}^b \rightarrow \mathcal{S}_{\gamma(t)}^b$  ( $s \leq t$ ) is defined by moving points of  $\mathcal{S}_{\gamma(s)}$  to  $\mathcal{S}_{\gamma(t)}$  along connection paths. In [4], it was proved that connection maps  $h_{st}$  are bilipschitz maps. For  $\mu \in \mathcal{MF}$  and  $\sigma \in \mathcal{T}$ , the length of  $\mu$  with respect to  $\sigma$  is defined by  $len_\sigma(\mu) = \int d\mu$ . From proposition 2.3, it follows that for any leaf segment  $l_s$  of  $\mu_s$ , the sequence of lengths  $len_{s+i}(h_{s,s+i}(l_s))$  satisfies the flaring property. As a consequence, we have the following theorem :

**Theorem 2.4.** (Lemma 3.6 of [9]) *For a compact subset  $\mathcal{B}$  of the moduli space  $\mathcal{M}$  and for every  $\rho \geq 1$ , there exist constants  $L \geq 1, \kappa > 1, n \in \mathbb{Z}_+$  such that the following holds: Let  $\gamma : I \rightarrow \mathcal{T}$  be a  $\mathcal{B}$ -cobounded and  $\rho$ -Lipschitz path, for any  $\mu \in \mathcal{M}$ , the sequence  $i \rightarrow len_{\gamma(i)}(\mu^b)$ , ( $i \in I \cap \mathbb{Z}$ ), satisfies the  $L$ -Lipschitz,  $(\kappa, n, 0)$ -flaring property.*

For  $\mu \in \mathcal{MF}$ , we say  $\mu$  is *realized* at  $p$ , where  $p$  is a finite number or  $p \in \{-\infty, +\infty\}$ , if  $len_{\gamma(i)}(\mu)$  achieves minimum at  $p$ .

**Proposition 2.5.** (Proposition 3.12 of [9]) *For each  $k \in I \cap \mathbb{Z}$ , there exists  $\mu \in \mathcal{MF}$  which is finitely realized. If  $I$  is infinite, for each infinite end  $\pm\infty$  of  $I$  there exists  $\mu_\pm \in \mathcal{MF}$  which is realized at  $\pm\infty$  respectively.*

Now for a compact subset  $\mathcal{B} \subset \mathcal{M}$ , numbers  $\rho \geq 1, \delta \geq 0, \eta > 0$ , consider  $\Gamma_{\beta, \rho, \delta, \eta}$  to be the set of all triples  $(\gamma, \mu_-, \mu_+)$  with the following properties (see [9]):

- (1)  $\gamma : I \rightarrow \mathcal{T}$  is  $\mathcal{B}$ -cobounded,  $\rho$ -Lipschitz path, such that  $\mathcal{H}_\gamma$  is  $\delta$ -hyperbolic relative to  $\mathcal{D}_\gamma$ ,
- (2)  $0 \in I$ , and each  $\mu_\pm \in \mathcal{MF}$  is normalized to have length 1 in the hyperbolic structure  $\gamma(0)$ ,
- (3) the lamination  $\mu_+$  is realized in  $\mathcal{S}_\gamma$  near the right end in the following way:
  - (a) If  $I$  is right infinite, then  $\mu_+$  is realized at  $+\infty$ ,
  - (b) If  $I$  is right finite, with right end point  $M$ , then there exists a minimum of length sequence  $len_{\gamma(i)}(\mu_+)$  lying in the interval  $[M - \eta, M]$ .

The lamination  $\mu_-$  is realized similarly in  $\mathcal{S}_\gamma$  near the left end.

Let  $\mathcal{A} \subset \mathcal{T}$  be a compact set such that each  $(\gamma, \mu_-, \mu_+) \in \Gamma_{\beta, \rho, \delta, \eta}$ , may be translated by the action of  $MCG(S_{g,n})$  so that  $\gamma(0) \in \mathcal{A}$ . If  $\gamma_i$  converges to  $\gamma$ , then in the Gromov-Hausdorff topology,  $\mathcal{H}_{\gamma_i}$  converges to  $\mathcal{H}_\gamma$  and  $\mathcal{D}_{\gamma_i}$  converges to  $\mathcal{D}_\gamma$ . Hence,  $\mathcal{G}(\mathcal{H}_{\gamma_i}, \mathcal{D}_{\gamma_i})$  converges to  $\mathcal{G}(\mathcal{H}_\gamma, \mathcal{D}_\gamma)$  in the Gromov-Hausdorff topology. The Gromov-Hausdorff limit of a sequence of  $\delta$ -hyperbolic spaces is  $\delta$ -hyperbolic ([5]). Therefore, if  $\mathcal{H}_{\gamma_i}$  are  $\delta$ -hyperbolic relative to  $\mathcal{D}_{\gamma_i}$  for all  $i$ , then  $\mathcal{H}_\gamma$  is also  $\delta$ -hyperbolic relative to  $\mathcal{D}_\gamma$ . This justifies the set  $\mathcal{A}_{\beta, \rho, \delta, \eta} = \{(\gamma, \mu_-, \mu_+) \in \Gamma_{\beta, \rho, \delta, \eta} : \gamma(0) \in \mathcal{A}\}$  is compact.

**Proposition 2.6.** [9] *The action of  $MCG(S_{g,n})$  on  $\Gamma_{\beta, \rho, \delta, \eta}$  is cocompact.*

### Proof of Theorem 2.1

For  $(\gamma, \mu_-, \mu_+) \in \Gamma_{\beta, \rho, \delta, \eta}$ , let  $a_-(t) = \frac{1}{len_{\gamma(t)}(\mu_-)}$  and  $a_+(t) = \frac{1}{len_{\gamma(t)}(\mu_+)}$ .  $\mu_-, \mu_+$  fills  $S_{g,n}$  (See [9]), therefore  $\mu_-, \mu_+$  defines a conformal structure  $\sigma(\mu_-, \mu_+)$  on  $S_{g,n}$ . Consider the map  $\xi(t) = \sigma(a_-(t)\mu_-, a_+(t)\mu_+)$ ,  $t \in I$ , then the image of the map  $\xi : I \rightarrow \mathcal{T}$  is a geodesic in  $\mathcal{T}$  joining  $\mu_-$  and  $\mu_+$ . For  $i \in I \cap \mathbb{Z}$ , define  $\gamma'(s) = \gamma(s+i)$ , then the triple  $(\gamma', a_-(t)\mu_-, a_+(t)\mu_+)$  lies in a translate of the compact set  $\mathcal{A}_{\beta, \rho, \delta, \eta}$  by an element of  $MCG(S_{g,n})$ . The map taking  $(\alpha, \lambda_-, \lambda_+) \in \Gamma_{\beta, \rho, \delta, \eta}$  to  $(\alpha(0), \sigma(\lambda_-, \lambda_+)) \in \mathcal{T} \times \mathcal{T}$  is  $MCG(S_{g,n})$  equivariant and continuous and

hence has  $MCG(S_{g,n})$  cocompact image. Therefore, the Teichmuller distance  $d_{\mathcal{T}}$  between  $\gamma(i)$  and  $\sigma(a_-(i)\mu_-, a_+(i)\mu_+) = \xi(i)$  is bounded. Now for  $t \in I$ , there exists  $i \in I \cap \mathbb{Z}$  such that  $|t - i| \leq 1$ . As  $\gamma$  is  $\rho$ -Lipschitz, therefore  $d_{\mathcal{T}}(\gamma(t), \gamma(i)) \leq \rho$ . Also, there exists  $L > 0$  such that  $d_{\mathcal{T}}(\xi(t), \xi(i)) \leq L$  (See [9]). Thus, there exists  $P > 0$  such that the Hausdorff distance between  $\gamma$  and  $\xi$  is at most  $P$ .  $\square$

### 3. APPLICATION

Consider the following short exact sequence of pair of finitely generated groups:

$$1 \rightarrow (\pi_1(S_{g,1}), K_1) \rightarrow (G, N_G(K_1)) \rightarrow (Q, Q) \rightarrow 1,$$

where  $K_1$  is peripheral subgroup of  $\pi_1(S_{g,1})$ ,  $G$  is strongly hyperbolic relative to  $N_G(K_1)$  and  $Q$  is a subgroup of  $MCG(S_{g,1})$ . Let  $\Phi : Q \rightarrow \mathcal{T}$  denote the orbit map, then for any geodesic  $\gamma' : I \rightarrow Q$ ,  $\gamma = \Phi \circ \gamma' : I \rightarrow \mathcal{T}$  is a cobounded and Lipschitz path. Since  $G$  is strongly hyperbolic relative to  $N_G(K_1)$ , the bundle  $\mathcal{E}(G, K_1)$  over  $Q$  is hyperbolic. Hence,  $\mathcal{E}(G, K_1) \rightarrow Q$  satisfies flaring property. In particular, the sub-bundle  $\mathcal{PH}_{\gamma} \rightarrow I$  satisfies the flaring property. Therefore, by the converse of strong combination theorem in [6],  $\mathcal{H}_{\gamma}$  is strongly hyperbolic relative to  $\mathcal{D}_{\gamma}$ . Hence, as an application of Theorem 2.1,  $Q$  is a convex cocompact subgroup of  $MCG(S_{g,1})$ . The converse of this result is also true (see [8]). So, we have the following theorem :

**Theorem 3.1.** [8] *Consider the following short exact sequence of pair of finitely generated groups*

$$1 \rightarrow (\pi_1(S_{g,1}), K_1) \rightarrow (G, N_G(K_1)) \rightarrow (Q, Q) \rightarrow 1,$$

*where  $\pi_1(S_{g,1})$  is strongly hyperbolic relative to  $K_1$ .  $G$  is strongly hyperbolic relative to  $N_G(K_1)$  if and only if  $Q$  is a convex cocompact subgroup of  $MCG(S_{g,1})$*

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